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A Grothendieck-type theorem for the space of totally measurable functions

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ABSTRACT

Let Σ be a σ-algebra of subsets of a non-empty set Ω. Let X be a real Banach space and let X^* stand for the Banach dual of X. Let $B(\Sigma, X)$ be the Banach space of Σ-totally measurable functions $f:\Omega \to X$, and let $B(\Sigma, X)^*$ and $B(\Sigma, X)^{**}$ denote the Banach dual and the Banach bidual of $B(\Sigma, X)$ respectively. Let $\mathrm{bvca}(\Sigma, X^*)$ denote the Banach space of all countably additive vector measures $\nu:\Sigma \to X^*$ of bounded variation. We prove a form of generalized Vitali–Hahn–Saks theorem saying that relative $\sigma(\mathrm{bvca}(\Sigma, X^*), B(\Sigma, X))$-sequential compactness in $\mathrm{bvca}(\Sigma, X^*)$ implies uniform countable additivity. We derive that if X reflexive, then every relatively $\sigma(B(\Sigma, X)^*, B(\Sigma, X))$-sequentially compact subset of $B(\Sigma, X)_c^\sim$ (= the σ-order continuous dual of $B(\Sigma, X)$) is relatively $\sigma(B(\Sigma, X)^*, B(\Sigma, X)^{**})$-sequentially compact. As a consequence, we obtain a Grothendieck type theorem saying that $\sigma(B(\Sigma, X)^*, B(\Sigma, X))$-convergent sequences in $B(\Sigma, X)_c^\sim$ are $\sigma(B(\Sigma, X)^*, B(\Sigma, X)^{**})$-convergent.

1. INTRODUCTION AND PRELIMINARIES

Let $(X, \|\cdot\|_X)$ be real Banach spaces and let B_X stand for the closed unit ball in X. Let X^* stand for the Banach dual of X. We denote by $\sigma(L, K)$ the weak topology with respect to a dual pair $\langle L, K\rangle$. For terminology concerning vector lattices we refer to [1]. By $\mathbb{N}$ and $\mathbb{R}$ we denote the sets of natural and real numbers.

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Let Σ be a σ-algebra of subsets of a non-empty set Ω. Let $\mathbb{1}_A$ denote the characteristic function of a set $A \in \Sigma$. By $\mathcal{S}(\Sigma, X)$ we denote the space of all X-valued Σ-simple functions $s = \sum_{i=1}^n \mathbb{1}_{A_i} \otimes x_i$, where $(A_i)_{i=1}^n$ is a Σ-partition of Ω, $x_i \in X$ for $1 \leqslant i \leqslant n$, and $(\mathbb{1}_{A_i} \otimes x_i)(\omega) = \mathbb{1}_{A_i}(\omega)x_i$ for $\omega \in \Omega$. Let $B(\Sigma, X)$ denote the Banach space of all Σ-totally measurable functions $f : \Omega \to X$ (the uniform limits of sequences of X-valued Σ-simple functions) provided with the supremum norm $\|\cdot\|_\infty$ (see [6, Section 6]). By $B(\Sigma, X)^*$ and $B(\Sigma, X)^{**}$ we will denote the Banach dual and the Banach bidual of $B(\Sigma, X)$ respectively. In case $X = \mathbb{R}$ we will simply write $B(\Sigma)$.

Denote by $\mathrm{bva}(\Sigma, X^*)$ the linear space of all vector measures $\nu : \Sigma \to X^*$ of bounded variation, i.e., $|\nu|(\Omega) < \infty$. If we equip $\mathrm{bva}(\Sigma, X^*)$ with the variation norm $\|\nu\| := |\nu|(\Omega)$, then $\mathrm{bva}(\Sigma, X^*)$ is a Banach space. It is known that $B(\Sigma, X)^*$ can be identified with $\mathrm{bva}(\Sigma, X^*)$ through the linear mapping: $\mathrm{bva}(\Sigma, X^*) \ni \nu \mapsto \Phi_\nu \in B(\Sigma, X)^*$, where

$$\Phi_\nu(f) = \int_\Omega f(\omega)\, d\nu \quad \text{for all } f \in B(\Sigma, X),$$

and $\|\Phi_\nu\| = |\nu|(\Omega)$ (see [6, Section 9, Corollary 1, p. 148]).

For $f \in B(\Sigma, X)$ let $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Then $\widetilde{f} \in B(\Sigma)$. Recall that a sequence (u_n) in $B(\Sigma)$ is order convergent to $u \in B(\Sigma)$ (in symbols $u_n \xrightarrow{(o)} u$) if there exists a sequence (v_n) in $B(\Sigma)$ such that $|u_n - u| \leqslant v_n$ holds for all $n \in \mathbb{N}$ and $v_n \downarrow 0$ in $B(\Sigma)$ (see [1, Definition 1.5]).

Now we can define σ-order continuous functionals on $B(\Sigma, X)$ (see [8]).

Definition. A linear functional $\Phi : B(\Sigma, X) \to \mathbb{R}$ is said to be *σ-order continuous* whenever $\Phi(f_n) \to 0$ for each sequence (f_n) in $B(\Sigma, X)$ such that $\widetilde{f} \xrightarrow{(o)} 0$ in $B(\Sigma)$. By $B(\Sigma, X)_c^\sim$ we will denote the set of all σ-order continuous linear functionals on $B(\Sigma, X)$. It is known that $B(\Sigma, X)_c^\sim \subset B(\Sigma, X)^*$ (see [8, Proposition 1.1]). For each $\omega \in \Omega$ and $x^* \in X^*$ let $\delta_{\omega,x^*}(f) = x^*(f(\omega))$ for all $f \in B(\Sigma, X)$. One can observe that $\delta_{\omega,x^*} \in B(\Sigma, X)_c^\sim$, so $B(\Sigma, X)_c^\sim$ separates the points of $B(\Sigma, X)$.

Let $\mathrm{bvca}(\Sigma, X^*)$ stand for the closed linear subspace of the Banach space $\mathrm{bva}(\Sigma, X^*)$ consisting of all those $\nu \in \mathrm{bva}(\Sigma, X^*)$ which are countably additive (see [5, Chapter 1, p. 30]). Then the space $B(\Sigma, X)_c^\sim$ can be identified with $\mathrm{bvca}(\Sigma, X^*)$ through the mapping: $\mathrm{bvca}(\Sigma, X^*) \ni \nu \mapsto \Phi_\nu \in B(\Sigma, X)_c^\sim$ (see [8, Corollary 3.2]). In particular, $\mathrm{bvca}(\Sigma, \mathbb{R}) = \mathrm{ca}(\Sigma)$ (= the space of all countably additive scalar measures). By $\mathrm{bva}(\Sigma, X^*)^*$ and $\mathrm{bvca}(\Sigma, X^*)^*$ we will denote the Banach duals of $\mathrm{bva}(\Sigma, X^*)$ and $\mathrm{bvca}(\Sigma, X^*)$, respectively.

Schaefer and Zhang ([9,11]) proved that if $\mathcal{K}$ is a $\sigma(B(\Sigma)^*, B(\Sigma))$-compact subset of $B(\Sigma)_c^\sim$ ($\cong \mathrm{ca}(\Sigma)$), then $\mathcal{K}$ is also $\sigma(B(\Sigma)^*, B(\Sigma)^{**})$-compact and the subset of $\mathrm{ca}(\Sigma)$ of corresponding measures is uniformly countably additive.

The aim of this paper is to extend these results to the case of the space $B(\Sigma, X)$. In Section 2, we prove a form of generalized Vitali–Hahn–Saks theorem saying that relative $\sigma(\mathrm{bvca}(\Sigma, X^*), B(\Sigma, X))$-sequential compactness in $\mathrm{bvca}(\Sigma, X^*)$ implies uniform countable additivity (see Theorem 2.1). In Section 3 we show that if X is a reflexive Banach space, then every relatively $\sigma(B(\Sigma,X)^*, B(\Sigma,X))$-sequentially compact subset of $B(\Sigma,X)^{\sim}_c$ is relatively $\sigma(B(\Sigma,X)^*, B(\Sigma,X)^{**})$-sequentially compact. As a consequence, we obtain a Grothendieck type theorem saying that $\sigma(B(\Sigma,X)^*, B(\Sigma,X))$-convergent sequences in $B(\Sigma,X)^{\sim}_c$ are $\sigma(B(\Sigma,X)^*, B(\Sigma,X)^{**})$-convergent.

2. A GENERALIZED VITALI–HAHN–SAKS TYPE THEOREM FOR $\mathrm{bvca}(\Sigma, X^*)$

We start with the main result of this section.

Theorem 2.1. *Let $\mathcal{M}$ be a relatively $\sigma(\mathrm{bvca}(\Sigma,X^*), B(\Sigma,X))$-sequentially compact subset of $\mathrm{bvca}(\Sigma,X^*)$. Then the following statements hold:*

(i) $\sup_{\nu\in\mathcal{M}} |\nu|(\Omega) < \infty$.
(ii) *$\{|\nu| : \nu\in\mathcal{M}\}$ is a uniformly countably additive subset of $\mathrm{ca}^+(\Sigma)$.*
(iii) *For each $A\in\Sigma$ the set $\{\nu(A) \colon \nu\in\mathcal{M}\}$ is relatively $\sigma(X^*, X)$-sequentially compact in X^*.*

Proof. (i) Since $\mathcal{M}$ is $\sigma(\mathrm{bvca}(\Sigma,X^*), B(\Sigma,X))$-bounded (see [10, Problem 6-4-106, p. 86]), the set $\{\Phi_\nu \colon \nu\in\mathcal{M}\}$ is $\sigma(B(\Sigma,X)^{\sim}_c, B(\Sigma,X))$-bounded. Hence by the Banach–Steinhaus theorem, we get $\sup_{\nu\in\mathcal{M}} |\nu|(\Omega) = \sup_{\nu\in\mathcal{M}} \|\Phi_\nu\| < \infty$.

(ii) Assume on the contrary that (ii) does not hold. Then in view of [4, Theorem 7.10] and the Rosenthal lemma (see [4, Chapter 7, p. 82]) there exist a pairwise disjoint sequence (A_n) in Σ, a positive number ε_0 and a sequence (ν_n) in $\mathcal{M}$ such that

$$(1)\qquad |\nu_n|(A_n) > \varepsilon_0 \quad\text{and}\quad |\nu_n|\Big(\bigcup_{j\neq n} A_j\Big) < \frac{1}{8}\varepsilon_0 \quad \text{for all } n\in\mathbb{N}.$$

In view of (1) for each $n\in\mathbb{N}$ there exists a Σ-partition $(A_{n,i})_{i=1}^{i_n}$ of A_n such that $\sum_{i=1}^{i_n} \|\nu_n(A_{n,i})\|_{X^*} > \varepsilon_0$. Next, for each $i = 1,\ldots,i_n$ there exists $x_{n,i}\in B_X$ such that

$$\nu_n(A_{n,i})(x_{n,i}) \geqslant \|\nu_n(A_{n,i})\|_{X^*} - \frac{1}{2^{i+1}}\varepsilon_0.$$

Let $g_n = \sum_{i=1}^{i_n} \mathbb{1}_{A_{n,i}} \otimes x_{n,i} \in \mathcal{S}(\Sigma,X)$. Then g_n vanishes off A_n with $\|g_n\|_\infty \leqslant 1$ and

$$\int_\Omega g_n(\omega)\,d\nu_n = \sum_{i=1}^{i_n} \nu_n(A_{n,i})(x_{n,i}) \geqslant \sum_{i=1}^{i_n} \|\nu_n(A_{n,i})\|_{X^*} - \sum_{i=1}^{i_n} \frac{1}{2^{i+1}}\varepsilon_0 \geqslant \frac{1}{2}\varepsilon_0.$$

Let (ν_{k_n}) be any subsequence of (ν_n), and let $g = \sum_{n=1}^{\infty} g_{k_{2n}}$ for all $n \in \mathbb{N}$. Clearly $g \in B(\Sigma, X)$ with $\|g\|_\infty \leqslant 1$, $g = g_{k_{2n}}$ on $A_{k_{2n}}$ and $g = 0$ on $A_{k_{2n+1}}$ for all $n \in \mathbb{N}$. Hence by (1) for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\int_\Omega g(\omega)\, d\nu_{k_{2n}} &= \int_{A_{k_{2n}}} g(\omega)\, d\nu_{k_{2n}} + \int_{\bigcup_{j \neq k_{2n}} A_j} g(\omega)\, d\nu_{k_{2n}} \\
&\geqslant \int_\Omega g_{k_{2n}}(\omega)\, d\nu_{k_{2n}} - |\nu_{k_{2n}}|\Bigl(\bigcup_{j \neq k_{2n}} A_j\Bigr) \\
&\geqslant \frac{1}{2}\varepsilon_0 - \frac{1}{8}\varepsilon_0 = \frac{3}{8}\varepsilon_0.
\end{aligned}$$

Moreover, using (1) we get for each $n \in \mathbb{N}$,

$$\begin{aligned}
\int_\Omega g(\omega)\, d\nu_{k_{2n+1}} &= \int_{\bigcup_{j=1}^{\infty} A_{k_{2j}}} g(\omega)\, d\nu_{k_{2n+1}} \\
&\leqslant \|g\|_\infty |\nu_{k_{2n+1}}|\Bigl(\bigcup_{j=1}^{\infty} A_{k_{2j}}\Bigr) \\
&\leqslant |\nu_{k_{2n+1}}|\Bigl(\bigcup_{j \neq k_{2n+1}} A_j\Bigr) < \frac{1}{8}\varepsilon_0.
\end{aligned}$$

This means that (ν_{k_n}) is not a $\sigma(\mathrm{bvca}(\Sigma, X^*), B(\Sigma, X))$-Cauchy sequence, because for $g \in B(\Sigma, X)$ the limit of $\langle \nu_{k_n}, g\rangle$ $(= \int_\Omega g(\omega)\, d\nu_{k_n})$ does not exist. It follows that $\mathcal{M}$ is not a $\sigma(\mathrm{bvca}(\Sigma, X^*), B(\Sigma, X))$-sequentially compact subset of $\mathrm{bvca}(\Sigma, X^*)$.

(iii) Note that for each $A \in \Sigma$ the mapping $\mathrm{bvca}(\Sigma, X^*) \ni \nu \mapsto \nu(A) \in X^*$ is $(\sigma(\mathrm{bvca}(\Sigma, X^*), B(\Sigma, X)), \sigma(X^*, X))$-continuous. Hence for each $A \in \Sigma$, the set $\{\nu(A)\colon \nu \in \mathcal{M}\}$ is relatively $\sigma(X^*, X)$-sequentially compact. $\square$

In view of Theorem 2.1 and [4, Theorem 7.13] we immediately obtain the following.

Corollary 2.2. *Let $\mathcal{M}$ be a relatively $\sigma(\mathrm{bvca}(\Sigma, X^*), B(\Sigma, X))$-sequentially compact subset of $\mathrm{bvca}(\Sigma, X^*)$. Then there exists $\mu \in \mathrm{ca}^+(\Sigma)$ such that $\lim_{\mu(A)\to 0} \sup_{\nu \in \mathcal{M}} |\nu|(A) = 0$.*

The following classical criterion for weak compactness in $\mathrm{bvca}(\Sigma, X^*)$ will be of importance (see [2, Corollary 1], [3, Proposition 3.1, p. 151]).

Theorem 2.3. *Assume that X is a reflexive Banach space. Then for a subset $\mathcal{M}$ of $\mathrm{bvca}(\Sigma, X^*)$ the following statements are equivalent:*

(i) *$\mathcal{M}$ is relatively $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathrm{bvca}(\Sigma, X^*)^*)$-compact.*
(ii) *$\mathcal{M}$ is relatively $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathrm{bvca}(\Sigma, X^*)^*)$-sequentially compact.*

(iii) $\sup_{\nu\in\mathcal{M}} |\nu|(\Omega) < \infty$ *and the set* $\{|\nu|\colon\ \nu \in \mathcal{M}\}$ *in* $\mathrm{ca}^+(\Sigma)$ *is uniformly countably additive.*

As a consequence of Theorems 2.1 and 2.3 we have the following corollary.

Corollary 2.4. *Assume that X is a reflexive Banach space. Then for a bounded subset $\mathcal{M}$ of the Banach space* $\mathrm{bvca}(\Sigma, X^*)$ *the following statements are equivalent:*

(i) *$\mathcal{M}$ is relatively $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathrm{bvca}(\Sigma, X^*)^*)$-compact.*
(ii) *$\mathcal{M}$ is relatively $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathrm{bvca}(\Sigma, X^*)^*)$-sequentially compact.*
(iii) *The set $\{|\nu|\colon\ \nu \in \mathcal{M}\}$ in $\mathrm{ca}^+(\Sigma)$ is uniformly countably additive.*
(iv) *$\mathcal{M}$ is relatively $\sigma(\mathrm{bvca}(\Sigma, X^*), B(\Sigma, X))$-sequentially compact.*

3. A GROTHENDIECK TYPE THEOREM FOR $B(\Sigma, X)$

For a subset $\mathcal{K}$ of $B(\Sigma, X)^*$ let $\mathcal{M}_{\mathcal{K}} = \{\nu \in \mathrm{bvca}(\Sigma, X^*)\colon\ \Phi_\nu \in \mathcal{K}\}$.

Theorem 3.1. *Assume that X is a reflexive Banach space. Then for a bounded subset of $\mathcal{K}$ of $B(\Sigma, X)^\sim_c$ the following statements are equivalent:*

(i) *$\mathcal{K}$ is relatively $\sigma(B(\Sigma, X)^*, B(\Sigma, X)^{**})$-compact.*
(ii) *$\mathcal{K}$ is relatively $\sigma(B(\Sigma, X)^*, B(\Sigma, X)^{**})$-sequentially compact.*
(iii) *$\mathcal{K}$ is relatively $\sigma(B(\Sigma, X)^*, B(\Sigma, X))$-sequentially compact.*
(iv) *The set $\{|\nu|\colon\ \nu \in \mathcal{M}_{\mathcal{K}}\}$ in $\mathrm{ca}^+(\Sigma)$ is uniformly countably additive.*

Proof. (i) $\Leftrightarrow$ (ii) It follows from the Eberlein theorem.

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (iv) Assume that $\mathcal{K}$ is relatively $\sigma(B(\Sigma, X)^*, B(\Sigma, X))$-sequentially compact. Since $B(\Sigma, X)^\sim_c$ is sequentially closed in $(B(\Sigma, X)^*, \sigma(B(\Sigma, X)^*, B(\Sigma, X)))$ (see [8, Corollary 2.3]), we conclude that $\mathcal{K}$ is relatively $\sigma(B(\Sigma, X)^\sim_c, B(\Sigma, X))$-sequentially compact subset of $B(\Sigma, X)^\sim_c$. Hence by Theorem 2.1 the set $\{|\nu|\colon\ \nu \in \mathcal{M}_{\mathcal{K}}\}$ is uniformly countably additive.

(iv) $\Rightarrow$ (i) Assume that the set $\{|\nu|\colon\ \nu \in \mathcal{M}_{\mathcal{K}}\}$ in $\mathrm{ca}^+(\Sigma)$ is uniformly countably additive. Then by Corollary 2.4 $\mathcal{M}_{\mathcal{K}}$ is a relatively $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathrm{bvca}(\Sigma, X^*)^*)$-compact subset of $\mathrm{bvca}(\Sigma, X^*)$. This means that $\mathcal{K}$ is a relatively compact set in $(B(\Sigma, X)^\sim_c, \sigma(\Sigma, X)^\sim_c, (B(\Sigma, X)^\sim_c)^*)$. Moreover, since $\mathrm{bvca}(\Sigma, X^*)$ is a closed subset of the Banach space $\mathrm{bva}(\Sigma, X^*)$, we have that $B(\Sigma, X)^\sim_c$ is a closed subset of the Banach space $B(\Sigma, X)^*$. Hence $B(\Sigma, X)^\sim_c$ is a closed set in $(B(\Sigma, X)^*, \sigma(B(\Sigma, X)^*,\ B(\Sigma, X)^{**}))$, so

$$\mathrm{cl}_{\sigma(B(\Sigma,X)^*,B(\Sigma,X)^{**})}\,\mathcal{K} \subset B(\Sigma, X)^\sim_c.$$

Note that

$$\sigma\big(B(\Sigma, X)^*, B(\Sigma, X)^{**}\big)|_{B(\Sigma,X)^\sim_c} = \sigma\big(B(\Sigma, X)^\sim_c, \big(B(\Sigma, X)^\sim_c\big)^*\big) \tag{2}$$

(see [7, Corollary 3.3.3]). It follows that

$$\text{(3)}\qquad \mathrm{cl}_{\sigma(B(\Sigma,X)_c^\sim,(B(\Sigma,X)_c^\sim)^*)}\mathcal{K} = \mathrm{cl}_{\sigma(B(\Sigma,X)^*,B(\Sigma,X)^{**})}\mathcal{K}.$$

Since $\mathrm{cl}_{\sigma(B(\Sigma,X)_c^\sim,(B(\Sigma,X)_c^\sim)^*)}\mathcal{K}$ is a $\sigma(B(\Sigma,X)_c^\sim,(B(\Sigma,X)_c^\sim)^*)$-compact subset of $B(\Sigma,X)_c^\sim$, in view of (2) and (3) we obtain that $\mathrm{cl}_{\sigma(B(\Sigma,X)^*,B(\Sigma,X)^{**})}\mathcal{K}$ is a $\sigma(B(\Sigma,X)^*,(B(\Sigma,X)^{**})$-compact subset of $B(\Sigma,X)^*$, i.e., $\mathcal{K}$ is relatively $\sigma(B(\Sigma,X)^*,B(\Sigma,X)^{**})$-compact, as desired. □

As a consequence of Theorem 3.1 we obtain that if X is reflexive, then $\sigma(B(\Sigma,X)^*, B(\Sigma,X))$-convergent sequences in $B(\Sigma,X)_c^\sim$ are $\sigma(B(\Sigma,X)^*, B(\Sigma,X)^{**})$-convergent.

Corollary 3.2. *Assume that X is a reflexive Banach space. Let $\nu_n \in \mathrm{bvca}(\Sigma,X^*)$ for $n\in\mathbb{N}$ and $\nu\in\mathrm{bvca}(\Sigma,X^*)$. Then the following statements are equivalent:*

(i) $\nu_n \xrightarrow{n} \nu$ *for* $\sigma(\mathrm{bva}(\Sigma,X^*),\mathrm{bva}(\Sigma,X^*)^*)$.
(ii) $\Phi_{\nu_n} \xrightarrow{n} \Phi_\nu$ *for* $\sigma(B(\Sigma,X)^*,B(\Sigma,X)^{**})$.
(iii) $\Phi_{\nu_n} \xrightarrow{n} \Phi_\nu$ *for* $\sigma(B(\Sigma,X)^*,B(\Sigma,X))$, *i.e.,* $\int_\Omega f(\omega)\,d\nu_n \xrightarrow{n} \int_\Omega f(\omega)\,d\nu$ *for each* $f\in B(\Sigma,X)$.

Moreover, if $\sup_n|\nu_n|(\Omega)<\infty$*, then each of the statements* (i)–(iii) *is equivalent to the following.*

(iv) $\nu_n(A)(x) \xrightarrow{n} \nu(A)(x)$ *for each* $A\in\Sigma$ *and* $x\in X$.

Proof. (i) ⇔ (ii) ⇒ (iii) It is obvious.

(iii) ⇒ (ii) Assume that $\Phi_{\nu_n}\xrightarrow{n}\Phi_\nu$ for $\sigma(B(\Sigma,X)^*,B(\Sigma,X))$ and let $(\Phi_{\nu_{k_n}})$ be a subsequence of (Φ_{ν_n}). Then $\mathcal{K}=\{\Phi_{\nu_{k_n}}: n\in\mathbb{N}\}$ is a relatively sequentially compact subset of $(B(\Sigma,X)^*,\sigma(B(\Sigma,X)^*,B(\Sigma,X)))$. By Theorem 3.1 $\mathcal{K}$ is a relatively sequentially compact subset of $(B(\Sigma,X)^*,\sigma(B(\Sigma,X)^*,B(\Sigma,X)^{**}))$, so there exists a subsequence $(\Phi_{\nu_{l_{k_n}}})$ of $(\Phi_{\nu_{k_n}})$ such that $\Phi_{\nu_{l_{k_n}}}\xrightarrow{n}\Phi_\nu$ for $\sigma(B(\Sigma,X)^*,B(\Sigma,X)^{**})$. This means that $\Phi_{\nu_n}\xrightarrow{n}\Phi_\nu$ for $\sigma(B(\Sigma,X)^*,B(\Sigma,X)^{**})$, as desired.

(iii) ⇒ (iv) It is obvious.

(iv) ⇒ (iii) Assume that (iv) holds, i.e., $\Phi_{\nu_n}(\mathbb{1}_A\otimes x)\xrightarrow{n}\Phi_\nu(\mathbb{1}_A\otimes x)$ for each $A\in\Sigma$ and $x\in X$. Since $\mathrm{cl}_{\|\cdot\|_\infty}\mathcal{S}(\Sigma,X)=B(\Sigma,X)$ and $\sup_n\|\Phi_{\nu_n}\|=\sup_n|\nu_n|(\Omega)<\infty$, by the Banach–Steinhaus theorem we conclude that $\Phi_{\nu_n}(f)\xrightarrow{n}\Phi_\nu(f)$ for all $f\in B(\Sigma,X)$. □

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